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Hyper-trigonometry of the particle triangle

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Abstract
Using a new kinematical description of a free three-body problem in hyperspherical coordinates (Matveenko A V and Fukuda H 1998 *J. Phys. A: Math. Gen.* **31** 5371) we derive two infinite series of matrix identities interconnecting triangle angles, particle masses and internal hyperspherical angles. The corresponding relations for the matrix elements are practically all new.

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1. Introduction

The hyperspherical harmonic (HH) method for three-body systems is, nowadays, at a very advanced level, though mainly in numerical applications (see Rosati and Viviani 1999, Jensen *et al* 1999 or Krivec and Mandelzweig 1999 for examples). More recently, we have suggested making use of a special minimal subset of HHs as a pure rotational part of primitives used in the variational treatment of three-body states. These 'physical' HHs proved to be useful as an instrument for the derivation of a new representation for the Wigner rotation matrices (Matveenko 1999) and identities including the associated Legendre polynomials of the same angle (Matveenko and Fukuda 1998).

In this paper, using the same idea, we derive matrix relations between HH solution matrices in different Jacobi channels (basically associated Legendre polynomials depending on different internal hyperspherical angles), with two more matrices being involved: that of the Raynal–Revai transformation (depending on the masses of the particles) and that of the Wigner rotation (depending on the internal triangle angle). The pilot variational calculations of some three-body Coulomb systems using these identities numerically (not discussing them) were presented at the *International Workshop on Resonances in Few-Body Systems (Sarospatak, 2000)*. The proceedings will appear soon (Matveenko *et al* 2001).

We give a simplest example of our identities in the introduction. Section 2 proceeds with mathematics needed for the derivation of two general identities presented in section 3. The content of the third section is completely new. Section 4 contains our conclusions.

For a system of three particles with masses $m_i (i = 1, 2, 3)$ we have three sets of Jacobi vectors $\{\boldsymbol{x}_i, \boldsymbol{y}_i\}$. As the basis in the particle plane we choose the set $\{i = 3\}$: the first Jacobi coordinate $\boldsymbol{x}_3 = \boldsymbol{x}$ to be the vector from particle 2 to particle 1, with the reduced mass $M_3 = M$; and the second Jacobi coordinate $\boldsymbol{y}_3 = \boldsymbol{y}$ from the centre of mass of $(1+2)$ to particle 3, with the reduced mass $\mu_3 = \mu$. For the reduced masses in the $\{i\}$-channel we have the well known expressions

$$\frac{1}{M_i} = \frac{1}{m_j} + \frac{1}{m_k} \qquad \frac{1}{\mu_i} = \frac{1}{m_i} + \frac{1}{m_j + m_k}. \tag{1}$$

Accordingly, three mass parameters, μ, M and $\kappa = (m_2 - m_1)/(m_1 + m_2)$, will be basic in our approach; using these the $\{i = 2\}$ Jacobi pair can be found from the equality

$$\begin{pmatrix} c_2 & 0 \\ 0 & 1/c_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_2 \\ \boldsymbol{y}_2 \end{pmatrix} = \begin{pmatrix} -\cos\phi_{23} & -\sin\phi_{23} \\ \sin\phi_{23} & -\cos\phi_{23} \end{pmatrix} \begin{pmatrix} c_3 & 0 \\ 0 & 1/c_3 \end{pmatrix} \begin{pmatrix} \boldsymbol{x}_3 \\ \boldsymbol{y}_3 \end{pmatrix} \tag{2}$$

where

$$c_2^4 = 4c/\rho_2^2 \qquad c_3^4 = 1/4c \qquad \sin^2\phi_{23} = 1/\rho_2$$

with $c = \mu/4M$ and $\rho_2 = 1 + c(1+\kappa)^2$. The transformation (2) includes the diagonal matrices of mass weighting and the orthogonal matrix of the so-called kinematic rotation by the angle ϕ_{23} (Raynal and Revai 1970).

Starting with the kinetic energy operator of a three-body problem in the centre-of-mass system we note that it is remarkably simple in hyperspherical coordinates

$$T = -\frac{1}{2M}\frac{1}{R^5}\frac{\partial}{\partial R}R^5\frac{\partial}{\partial R} + \frac{\Lambda^2}{2MR^2}. \tag{3}$$

Here, for our choice of the Jacobi basic channel $\{i = 3\}$ the hyperradius R will be defined by

$$R = \left(x^2 + \frac{\mu}{M}y^2\right)^{1/2}. \tag{4}$$

The squared hyperangular momentum operator Λ^2 and its eigenfunctions (HHs) defined by

$$\left(\Lambda^2 - K(K+4)\right)\boldsymbol{Y}_K(\Omega) = 0 \tag{5}$$

depend on five hyperangles Ω (Avery 1989). In this definition only the value of the grand angular momentum $K = 0, 1, 2\ldots$ has been specified so far. As Avery (1989) has found, the total number of degenerate HHs having the same value of K is

$$\dim(6, K) = (K+1)(K+2)^2(K+3)/6.$$

If HHs from different Jacobi channels are to be used in the applications we usually need the Raynal–Revai orthogonal transformation matrix that interconnects degenerate sets for a given value of K.

In our approach, we firstly make use of the fact that, if written in the body-fixed frame, HHs can be factorized into the extrinsic part depending on three Euler-rotation angles and the intrinsic one that depends on two internal variables

$$\boldsymbol{Y}_{KlL}^{JpM_J}(\alpha_i, \theta_i, \tilde{\gamma}, \tilde{\beta}, \tilde{\alpha}) = \sum_{m'=0(1)}^{J} y_{KlLm'}^{Jp}(\alpha_i, \theta_i) B_{m'}^{JpM_J}(\tilde{\gamma}, \tilde{\beta}, \tilde{\alpha}) \tag{6}$$

where the regular hyperspherical angles are defined by

$$\cos\theta_i = (\hat{\boldsymbol{x}}_i \hat{\boldsymbol{y}}_i) \qquad \tan\alpha_i = \frac{M_i^{1/2}\, x_i}{\mu_i^{1/2}\, y_i}. \tag{7}$$

Now, all quantum numbers needed for our discussion of the free nonrelativistic state are explicitly given, including those of the total parity p, total angular momentum J, with M_J and m' being projections of $\boldsymbol{J}$, and the usual angular momentum $\boldsymbol{l} = -\mathrm{i}\boldsymbol{y} \times \nabla_{\boldsymbol{y}}$, $\boldsymbol{L} = -\mathrm{i}\boldsymbol{x} \times \nabla_{\boldsymbol{x}}$. The body-fixed z-axis is specified by the set $\{\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}\}$, and $B_{m'}^{JpM_J}$ are the parity preserving combinations of the Wigner D-functions (Matveenko and Fukuda 1996).

In what follows we shall manipulate with two $\{i = 2\}$ and $\{j = 3\}$ 'physical' (Matveenko and Fukuda 1998) subsets of HHs satisfying either the $K = J$ or the $J + 1$ assumption, being equivalent to the conditions introduced by Schwartz (1961)

$$L + l = J \quad \text{if} \quad \{p = (-)^J\} \qquad \text{or} \qquad L + l = J + 1 \quad \text{if} \quad \{p = -(-)^J\}. \tag{8}$$

Actually, we shall use only intrinsic harmonics (IHHs), i.e. vector columns $\|y_{KlLm'}(\alpha_i, \theta_i)\|^{Jp}$, in which case, using (8), we can simplify the notation (quantum numbers K and L are now not needed). This means that we can write using $\epsilon = K - J$

$$\|y_{lm'}(\alpha_i, \theta_i)\|^{Jp} = \|y_{KlLm'}(\alpha_i, \theta_i)\|^{Jp} \qquad \epsilon \leqslant m' \leqslant l. \tag{9}$$

Here and below, though the running index of the vector column m' can be bigger than l the components of IHHs (9) for $m' > l$ are equal to zero, that allows one to arrange the solution matrices composed from (9) in the triangular form. The corresponding degeneracies of the specified subsets are easily found to be $N(K = J) = J + 1$ and $N(K = J + 1) = J$ for states of normal and abnormal parity, respectively, which makes all the solution matrices not only triangular but also square.

Say, using (2) one can form scalar products of vectors from different Jacobi sets which will provide the identities including the usual internal angle of the triangle $\theta'_{23} = \arccos(\hat{\boldsymbol{x}}_2\hat{\boldsymbol{x}}_3)$, the kinematic angle ϕ_{23} and the hyperspherical angles (7). We shall be able to derive two infinite series of matrix equalities of that kind, i.e. those interconnecting two pairs of hyperspherical angles (7). The typical example reads

$$\begin{aligned} &\begin{pmatrix} \cos\theta'_{23} & \sin\theta'_{23} \\ -\sin\theta'_{23} & \cos\theta'_{23} \end{pmatrix} \begin{pmatrix} \sin\alpha_2 & \cos\alpha_2\cos\theta_2 \\ 0 & -\cos\alpha_2\sin\theta_2 \end{pmatrix} \\ &\qquad = \begin{pmatrix} \sin\alpha_3 & \cos\alpha_3\cos\theta_3 \\ 0 & -\cos\alpha_3\sin\theta_3 \end{pmatrix} \begin{pmatrix} -\cos\phi_{23} & \sin\phi_{23} \\ -\sin\phi_{23} & -\cos\phi_{23} \end{pmatrix} \end{aligned} \tag{10}$$

or, as an example of the scalar identity,

$$\sin\phi_{23}\cos\alpha_3\sin\theta_3 = -\sin\alpha_2\sin\theta'_{23} \tag{11}$$

for the corresponding element of the matrix equation (10). The geometrical background of (10) is that the triangular solution matrix of vector-column intrinsic HHs in the channel $\{i = 2\}$ (utilizing $\hat{\boldsymbol{x}}_2$ as the quantization axis) (see below and the next section for more details), if multiplied from the left by the rotation matrix, is equal to the equivalent solution matrix but in the channel $\{i = 3\}$ (quantized with respect to $\hat{\boldsymbol{x}}_3$), multiplied from the right by the matrix of kinematic rotation. This is exactly the Raynal–Revai transformation relation used in the subspace of 'physical' HHs for their intrinsic part IHHs. An additional rotation matrix accounts for the different quantization axis used in the same equation (10).

Later we shall show that the matrix identity (10) is just the $(J = 1, p = -1)$ example of the general result interconnecting IHHs in $\{2\}$- and $\{3\}$-channels

$$\hat{\boldsymbol{d}}^{Jp}(\theta'_{23})\|\boldsymbol{p}(\alpha_2, \theta_2)\|^{Jp} = \|\boldsymbol{p}(\alpha_3, \theta_3)\|^{Jp}\mathcal{R}^{Jp}(\phi_{23}) \tag{12}$$

where the parity-projected Wigner $\hat{\boldsymbol{d}}^{Jp}(\theta)$ matrices are defined as by Matveenko (1999).

While the regular Raynal–Revai transformation allows one to interconnect five-dimensional HHs expressed in different Jacobi-channel coordinates, we relate (12) by the rotation in the particle plane two special degenerate subsets $(K = J)$ or $(K = J + 1)$ of the two-dimensional vector-column IHHs stemming from (6). It seems to be clear, but worth noting, that the derivation of (12) is a by-product of a formal quantum description of a free three-body problem in hyperspherical coordinates.

2. Intrinsic hyperspherical harmonics

In order to find the equation for the vector-column IHHs (9) one should search for the eigenfunctions of Λ^2 from the kinetic energy operator (3) in the form (6) and integrate over α, β, γ. This familiar procedure (actually, the partial wave analysis) will convert Λ^2 into the matrix operator $[\Lambda_i^2]^{Jp}$ and will result in the corresponding system of the Schrödinger equations in two variables (see below). In our approach we should not specify the position of the body-fixed quantization axis; let us just mention that the resulting equations will strongly depend on its location (Matveenko and Fukuda 1996).

Using the auxiliary vector column $\|p_l(\alpha_i, \theta_i)\|^{Jp}$ for all possible $l = \epsilon, \dots, J$

$$\|p_l(\alpha_i, \theta_i)\|^{Jp} = \frac{\sin^L \alpha_i \cos^l \alpha_i}{\sqrt{l!!\, L!!}} \begin{pmatrix} (-1)^\epsilon U_{\epsilon L}^{Jpl}\ P_l^\epsilon(\theta_i) \\ \dots \\ (-1)^m U_{mL}^{Jpl}\ P_l^m(\theta_i) \\ \dots \\ (-1)^l U_{lL}^{Jpl}\ P_l^l(\theta_i) \\ 0 \\ \dots \\ 0 \end{pmatrix} \qquad m = \epsilon, \dots, l\{J\} \tag{13}$$

with the coefficients

$$U_{mL}^{Jpl} = p(-)^{J+l+m}\sqrt{2 - \delta_{0m}}(l, J, -m, m|L, 0) \qquad \left(\sum_{m=0}^{l}(U_{mL}^{Jpl})^2 = 1\right) \tag{14}$$

(Chang and Fano 1972), we define the basic entity of the approach: the vector-column IHH

$$\|y_l(\alpha_i, \theta_i)\|_\omega^{Jp} = \boldsymbol{d}^{Jp}(\omega_i)\ \|p_l(\alpha_i, \theta_i)\|^{Jp} \tag{15}$$

with an arbitrary choice of the quantization axis within the particle plain (direction $\boldsymbol{\omega}$). Here, $\cos\omega_i = \boldsymbol{\omega} \cdot \hat{\boldsymbol{x}}_i$, $\boldsymbol{d}^{Jp}(\omega_i)$ is the parity preserving combination of the Wigner rotation matrices (Matveenko 1999) and $P_l^m(\theta_i)$ are the normalized associated Legendre polynomials.

In accord with (5), the IHHs $\|y_l(\alpha_i, \theta_i)\|_\omega^{Jp}$ solve the matrix differential equation in two variables

$$\left([\Lambda_i^2]_\omega^{Jp} - K(K+4)\right) \|y_l(\alpha_i, \theta_i)\|_\omega^{Jp} = 0 \qquad \epsilon \leqslant l \leqslant J$$

where $[\Lambda_i^2]_\omega^{Jp}$ is the hyperangular part of the total three-body kinetic energy operator (3) projected onto the states of fixed total angular momentum J and parity p. If we make use of the special coordinate system with $\boldsymbol{\omega} = \hat{\boldsymbol{x}}_3$ we have $\omega_3 = 0$, $\hat{\boldsymbol{d}}^{Jp}(0)$ will be simply a unity matrix and we obtain using (15) the identity $\|y_l(\alpha_3, \theta_3)\|_{\omega=\hat{x}_3}^{Jp} = \|p_l(\alpha_3, \theta_3)\|^{Jp}$.

Finishing the section, we illustrate its content by writing down the details for the $(J = 1, p = -1)$ case. Since it is a normal parity case, we have two IHHs with $l = 0, 1$ in each Jacobi channel. The solution matrix composed from (15) for $\{i = 2, 3\}$ channels will be

$$\hat{\boldsymbol{d}}^{1,-1}(\theta'_{23}) \begin{pmatrix} \sin\alpha_2 & \cos\alpha_2\cos\theta_2 \\ 0 & -\cos\alpha_2\sin\theta_2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} \sin\alpha_3 & \cos\alpha_3\cos\theta_3 \\ 0 & -\cos\alpha_3\sin\theta_3 \end{pmatrix} \tag{16}$$

respectively. Here, for the $\{i = 3\}$ case no additional rotation is needed as $\boldsymbol{x} = \boldsymbol{x}_3$ was chosen as the quantization axis, and for the $\{i = 2\}$ channel the corresponding rotation matrix is (Matveenko 1999)

$$\hat{\boldsymbol{d}}^{1,-1}(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{17}$$

The eigenvalue equation utilizing the $\{i = 3\}$ solution matrix (16) will be

$$\left([\Lambda_3^2]^{1,-1}_{\hat{\omega}=\hat{x}_3} - 5\right)\begin{pmatrix} \sin\alpha_3 & \cos\alpha_3\cos\theta_3 \\ 0 & -\cos\alpha_3\sin\theta_3 \end{pmatrix} = 0 \tag{18}$$

with the following chain of notation:

$$[\Lambda_3^2]^{1,-1}_{\omega=\hat{x}_3} = \Lambda_3^2 - \frac{1}{\sin^2\alpha_3\cos^2\alpha_3}\left(\frac{1}{\sin\theta_3}\frac{\partial}{\partial\theta_3}\sin\theta_3\frac{\partial}{\partial\theta_3} - \frac{1}{\sin^2\theta_3}J_x^2\right) + \frac{\boldsymbol{J}^2 + \hat{B}}{\sin^2\alpha_3}$$

$$\Lambda_3^2 = -\frac{1}{\sin^2 2\alpha_3}\frac{\partial}{\partial\alpha_3}\sin^2 2\alpha_3\frac{\partial}{\partial\alpha_3}$$

$$\boldsymbol{J}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad J_x^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad \hat{B} = -\begin{pmatrix} 0 & \frac{\partial}{\partial\theta_3} + \cot\theta_3 \\ -\frac{\partial}{\partial\theta_3} & 0 \end{pmatrix}.$$

3. Hyper-trigonometry identities

There are only minor details that make the treatment of the normal and abnormal parity cases different. We shall start with the normal parity case: setting $p = (-)^J$ and using the $(p = n)$ index for the parity-dependent expressions. The body-fixed quantization axis $(\boldsymbol{\omega} = \hat{\boldsymbol{x}}_3 = \hat{\boldsymbol{x}})$ is still used in order to make the analysis simpler. As discussed above, the solution matrices in $i = \{2\}$ and $\{3\}$ Jacobi channels are $\|\boldsymbol{y}(\alpha_2, \theta_2)\|^{Jp}_{\hat{x}} = \hat{\boldsymbol{d}}^{Jp}(\theta_{23})\ \|\boldsymbol{p}(\alpha_2, \theta_2)\|^{Jp}$ and $\|\boldsymbol{y}(\alpha_3, \theta_3)\|^{Jp}_{\hat{x}} = \|\boldsymbol{p}(\alpha_3, \theta_3)\|^{Jp}$, respectively. Any IHH $\|y_l(\alpha_2, \theta_2)\|^{Jp}_{\hat{x}}$ can be expressed as a linear combination of $\|y_{l'}(\alpha_3, \theta_3)\|^{Jp}_{\hat{x}}$, $(\epsilon \leqslant l, l' \leqslant J)$; the corresponding transformation matrix R^{Jp} has been introduced by Raynal and Revai (1970) for five-dimensional HHs in the ordinary space-fixed frame. In our case, for vector-column IHHs depending only on two variables (15), the proper relation will be

$$\hat{\boldsymbol{d}}^{Jn}(\theta'_{23})\|\boldsymbol{p}(\alpha_2, \theta_2)\|^{Jn} = \|\boldsymbol{p}(\alpha_3, \theta_3)\|^{Jn}\|d^{J/2}_{-J/2+l,-J/2+l'}(2\phi_{23})\| \qquad (l, l' = 0, \dots, J) \tag{19}$$

where all involved matrices have the $(J + 1) * (J + 1)$ dimension. The upper triangle matrices $\|\boldsymbol{p}(\alpha_i, \theta_i)\|^{Jn}$ with matrix elements $p_{lm}(\alpha_i, \theta_i)$ have their columns (12) numbered by the quantum number of the pair angular momentum $l (0 \leqslant l \leqslant J)$ while its projection m onto the body-fixed $\boldsymbol{x}_i$-axis serves for row numbering (only $\|p_{lm}(\alpha_i, \theta_i)\|^{Jn}$ with $0 \leqslant m \leqslant l$ are nonzero). Two more matrices, normal parity rotation matrix $\hat{\boldsymbol{d}}^{Jn}$ and the usual Wigner rotation matrix $\|d^{J/2}_{-J/2+l,-J/2+l'}\|$, are orthogonal. We have used in (19) the result derived by Raynal (1973) allowing one to express Raynal–Revai matrices for the extreme value of the grand angular momentum $K = J$ in terms of $\mathcal{D}^{Jn}(\phi) = \|d^{J/2}_{-J/2+l,-J/2+l'}(2\phi)\|$, $(0 \leqslant l, l' \leqslant J)$. The coefficients (14), composing (12) for the normal parity case, read

$$U^{Jnl}_{mL} = \left(\frac{2}{1 + \delta_{m0}}\frac{(2l)!(2L + 1)!(J - m)!(J + m)!}{(2J + 1)!(l - m)!(l + m)!(L!)^2}\right)^{1/2} \qquad L = J - l. \tag{20}$$

Using the triangular structure of $\|\boldsymbol{p}(\alpha_i, \theta_i)\|^{Jn}$, and analytic expressions for the matrix elements of the first row (first column) of $\hat{\boldsymbol{d}}^{Jn}$ (Matveenko 1999) and $\|d^{J/2}_{m,m'}(\phi)\|$ (Varshalovich *et al*

1988), we present four simplest scalar identities for the corner matrix elements of the matrix equation (19). For the [0, 0] one (the upper left corner of (19)) we have

$$\frac{\sin^J \alpha_2}{\sqrt{(2J+1)J!!}} P_J^0(\theta'_{23}) = \sum_{l=0}^{J} \sqrt{\frac{J!}{l!(J-l)!}} p_{l0}(\alpha_3, \theta_3) \cos^{J-l} \phi_{23} \sin^l(-\phi_{23}). \tag{21}$$

The [0, J] case is the most complicated; it includes two summations

$$\sum_{m=0}^{J} \sqrt{\frac{2}{(1+\delta_{0m})(2J+1)}} P_m^J(-\theta'_{23})\ p_{Jm}(\alpha_2, \theta_2) = \sum_{l=0}^{J} \sqrt{\frac{J!}{l!(J-l)!}} p_{l0}(\alpha_3, \theta_3) \cos^l \phi_{23} \sin^{J-l} \phi_{23}. \tag{22}$$

The simplest case is the [J, 0] one: it does not include the summation, does not depend on J and thus coincides with the ($J = 1$, $p = -1$) example from the introduction (11). Moreover, the last [J, J] corner provides

$$\frac{2}{\sqrt{J!!(2J+1)}} \cos^J \alpha_3 \cos^J \phi_{23} P_J^J(\theta_3) = \sum_{m=0}^{J} d_{Jm}^{Jn}(\theta'_{23})\ p_{Jm}(\alpha_2, \theta_2). \tag{23}$$

It can be checked that by putting $J = 1$ in the last three equations one is reproducing the corresponding scalar equalities from matrix equation (10).

For the abnormal parity states, i.e. those defined by the conditions $K = J + 1$ and $p = -(-)^J$, we obtain similarly the matrix identity interconnecting IHHs in the $i = \{2\}$ and $\{3\}$ Jacobi channels

$$\hat{\boldsymbol{d}}^{Ja}(\theta'_{23}) \|\boldsymbol{p}(\alpha_2, \theta_2)\|^{Ja} = \|\boldsymbol{p}(\alpha_3, \theta_3)\|^{Ja} \|d^{(J-1)/2}_{-(J+1)/2+l, -(J+1)/2+l'}(2\phi_{23})\| \qquad (1 \leqslant l, l' \leqslant J) \tag{24}$$

where now we have to deal with the $(J * J)$ matrices having $1 \leqslant l, m \leqslant J$ numbering columns and rows of $\|\boldsymbol{p}(\alpha_i, \theta_i)\|^{Ja}$ (as in the normal parity case matrix elements are nonzero only for $1 \leqslant m \leqslant l$). Here (24), once again we have used the result of Raynal (1973), this time for $K = J+1$ $\mathcal{R}^{Ja}(\phi_{23})$ being given as the Wigner rotation matrix. The Chang–Fano coefficients are now expressed by

$$U_{mL}^{Jal} = 2Lm \left(2(2L+1) \frac{(2l-1)!(2L-1)!(J-m)!(J+m)!}{(2J+2)!(l-m)!(l+m)!(L!)^2} \right)^{1/2} \qquad L = J - l + 1. \tag{25}$$

The four simplest scalar identities for the corner matrix elements of (24) will read

$$[1, 1] \rightarrow \sqrt{\frac{6}{J(J+1)(2J+1)J!!}} \cos\alpha_2 \sin^J \alpha_2 \sin\theta_2 \frac{P_J^1(\theta'_{23})}{\sin\theta'_{23}} = -\sum_{l=1}^{J} \sqrt{\frac{(J-1)!}{(l-1)!(J-l)!}} p_{l1}(\alpha_3, \theta_3) \cos^{J-l} \phi_{23} \sin^{l-1}(-\phi_{23}) \tag{26}$$

$$[1, J] \rightarrow \sqrt{\frac{6}{J(J+1)(2J+1)}} \sum_{m=1}^{J} m \frac{P_m^J(\theta'_{23})}{P_1^1(\theta'_{23})} p_{Jm}(\alpha_2, \theta_2) = -\sum_{l=1}^{J} \sqrt{\frac{(J-1)!}{(l-1)!(J-l)!}} p_{l1}(\alpha_3, \theta_3) \cos^{l-1} \phi_{23} \sin^{J-l} \phi_{23} \tag{27}$$

$$[J, 1] \;\rightarrow\; \cos\alpha_2 \sin\alpha_2 \sin\theta_2 = \cos\alpha_3 \sin\alpha_3 \sin\theta_3 \tag{28}$$

$$[J, J] \;\rightarrow\; J\sqrt{\frac{6}{J(J+1)(2J+1)}}\cos^J\alpha_3 \sin\alpha_3 \frac{P_J^J(\theta_3)}{\sqrt{J!!}}\cos^{J-1}\phi_{23}$$
$$= \sum_{m=1}^{J} d_{Jm}^{Ja}(\theta'_{23})\; p_{Jm}(\alpha_2, \theta_2). \tag{29}$$

Once again, the simplest $[J, 1]$ result does not depend on J and can be thought of as a hyperspherical *sin*-theorem; in order to obtain the above expression (28) the identity (11) should be applied.

Both matrix equalities, (19) and (24), were checked numerically. For this purpose we have introduced hyperspheroidal coordinates $\xi = (x_1 + x_2)/x_3$, $\eta = (x_1 - x_2)/x_3$ (Matveenko and Fukuda 1996) and expressed in their terms all auxiliary variables entering the above identities:

$$\sin\alpha_3 = 1/\sqrt{\rho} \qquad \cos\theta_3 = -(\xi\eta - \kappa)\sqrt{c/(\rho - 1)}$$
$$\sin\alpha_2 = (\xi + \eta)\sqrt{c/(\rho\rho_2)}$$
$$\cos\theta_2 = -[c(1+\kappa)(\xi+\eta)^2 - \rho_2(1+\xi\eta)]/[\sqrt{\rho\rho_2}(\xi+\eta)\cos\alpha_2]$$
$$\cos\theta'_{23} = -(1+\xi\eta)/(\xi+\eta)$$

with $\rho = 1 + c(\xi^2 + \eta^2 - 2\kappa\xi\eta + \kappa^2 - 1)$; constants ρ_2, c, κ and kinematic angle ϕ_{23} were defined in the introduction. The channel-independent equality (28) can be easily checked analytically to give

$$\cos\alpha_i \sin\alpha_i \sin\theta_i = \frac{\sqrt{c(\xi^2 - 1)(1 - \eta^2)}}{\rho} \qquad (i = 1, 2, 3). \tag{30}$$

4. Conclusions

For any given value of the total angular momentum of a three-body system the underlying matrix expressions (19) and (24) provide $(J + 1)^2$ and J^2 scalar equalities, respectively. Roughly speaking, they are all new; only a few of them can be easily derived using standard trigonometry tools. We hope that in realistic three-body calculations, utilizing hyperspherical coordinates, our identities will be helpful. One example of this kind can be found in our recent discussion of the semianalytic description of the highly rotational states of antiprotonic He (Matveenko and Alt 2000). In that paper, we were able to resolve the Coriolis coupling analytically using the angular part of the variational primitives in the form (15). In this case, the matrix structure of the Schrödinger operator can be reduced to the calculation of the analytic function $\sigma_{ll'}^{ij}(\xi, \eta) = \sum p_{lm}(\alpha_i, \theta_i) p_{l'm'}(\alpha_j, \theta_j) d_{mm'}^{Jp}(\omega_{ij})$, with $\cos\omega_{ij} = \hat{\boldsymbol{x}}_i \cdot \hat{\boldsymbol{x}}_j$. As at the end of section 3, before starting calculations we introduce a pair of global hyperspherical angles ξ, η, common to all Jacobi channels. Actually, $\sigma_{ll'}^{ij}$ is the matrix element of the 'angular form-factor matrix' $\left[\|\boldsymbol{y}^i\|_\omega^{Jp}\right]^T \left[\|\boldsymbol{y}^j\|_\omega^{Jp}\right]$, which, owing to (19) or (24), can be calculated using either of the two alternative expressions

$$\left[\|\boldsymbol{y}^i\|_\omega^{Jp}\right]^T \left[\|\boldsymbol{y}^j\|_\omega^{Jp}\right] = \left[\|\boldsymbol{y}^i\|_{\hat{x}_i}^{Jp}\right]^T \hat{\boldsymbol{d}}^{Jp}(\theta'_{ij})\|\boldsymbol{p}^j)\|^{Jn} = \left[\|\boldsymbol{p}^i\|^{Jp}\right]^T \|\boldsymbol{p}^j\|^{Jp} \mathcal{R}^{Jp}(\phi_{ij}) \tag{31}$$

where the equality (15) for the IHH solution matrices $\|\boldsymbol{y}^i)\|_{\omega=\hat{x}_i}^{Jp} = \|\boldsymbol{p}^i\|^{Jp}$ has been used.

The choice of the variational primitives in a vector-column form is really giving a new kinematical description in the adiabatic approach to a three-body problem: the magnetic quantum number is summed out when the corresponding generalized eigenvalue problem is formulated and the number of adiabatic states is decreased by the factor J. This allows one to overcome partially numerical difficulties provoked by sharply peaked radial and angular non-adiabatic couplings in the avoided crossing regions (Matveenko *et al* 2001).

A peculiar feature of the identities (19) and (24) is that they relate Wigner rotation matrices of two types: regular ones $\|d^J_{mm'})\|$, as defined by Varshalovich *et al* (1988), and parity-projected ones $\hat{\boldsymbol{d}}^{Jp}$. The latter can be defined in the factorized form, recently derived in a similar context by Matveenko (1999) and by Manakov *et al* (2000) using a different technique.

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